Extinction for two parabolic stochastic PDE's on the lattice

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Abstract

It is well known that, starting with finite mass, the super-Brownian motion dies out in finite time. The goal of this article is to show that with some additional work, one can show finite time die-out for two types of systems of stochastic differential equations on the lattice $\mathbf{Z}^d$.

For our first system, let $1/2 \leq \gamma < 1$, and consider non-negative solutions of

$$\begin{aligned} du(t,x) &= \Delta u(t,x)dt + u^{\gamma}(t,x)dB_x(t), \ \ x \in \mathbf{Z}^d \\ u(0,x) &= u_0(x) \geq 0. \end{aligned}$$

Here Δ is the discrete Laplacian and $\{B_x : x \in \mathbf{Z}^d\}$ is a system of independent Brownian motions. We assume that u_0 has finite support. When $\gamma = 1/2$, the measure which puts mass $u(t,x)$ at x is a super-random walk and it is well-known that the process becomes extinct in finite time a.s. Finite-time extinction is known to be a.s. false if $\gamma = 1$. For $1/2 < \gamma < 1$, we show finite-time die-out by breaking up the solution into pieces, and showing that each piece dies in finite time. Unlike the superprocess case, these pieces will not in general evolve independently.

Our second example involves the mutually catalytic branching system of stochastic differential equations on $\mathbf{Z}^{\mathbf{d}}$, which was first studied in Dawson and Perkins [DP98].

$$\begin{aligned} dU_t(x) &= \Delta U_t(x)dt + \sqrt{U_t(x)V_t(x)}dB_{1,x}(t) \\ dV_t(x) &= \Delta V_t(x)dt + \sqrt{U_t(x)V_t(x)}dB_{2,x}(t) \\ U_0(x) &\geq 0 \\ V_0(x) &\geq 0. \end{aligned}$$

By using a somewhat different argument, we show that, depending on the initial conditions, finite time extinction of one type may occur with probability 0, or with probability arbitrarily close to 1.

1 Introduction

Recently, the Dawson-Watanabe process, or super-Brownian motion, has attracted great interest, and many fascinating properties have come to light. See Dawson [Daw93] for a survey. These results often rely on the multiplicative property of the process. This allows one to study the process as an infinite system of noninteracting particles, each with infinitesimal mass. However, it is often much more difficult to prove similar results for systems with interactions.

In this article, we concentrate on the finite time extinction property. Let Z_t be the total mass of the Dawson-Watanabe process, and assume that the initial mass $Z_0 < \infty$. As is well known, Z_t satisfies the Feller equation

$$dZ = \sqrt{Z}dB \tag{1.1}$$

and with probability 1, Z_t reaches 0 in finite time. See Theorem 4.3.6 of [Kni81] for the exact extinction probabilities. Our goal is to study finite time extinction for 2 types of systems of stochastic differential equations (SDE), related to super-random walks, on the lattice $\mathbf{Z}^d$.

First, we consider non-negative solutions $u(t,x)$, $t \geq 0, x \in \mathbf{Z}^d$ to the following system of stochastic differential equations on the lattice $\mathbf{Z}^d$, for $1/2 \leq \gamma < 1$.

$$\begin{aligned} du(t,x) &= \Delta u(t,x)dt + u^{\gamma}(t,x)dB_x(t), \ x \in \mathbf{Z}^d \\ u(0,x) &= u_0(x) \geq 0. \end{aligned} \tag{1.2}$$

Here and throughout the paper, Δ is the discrete Laplacian on $\mathbf{Z}^d$. In other words, if $\mathcal{N}(x)$ is the set of $2d$ nearest neighbors of $x \in \mathbf{Z}^d$, and if $f(x)$ is a function on $\mathbf{Z}^d$, then $(\Delta f)(x) = \sum_{y\in\mathcal{N}(x)} f(y) - 2df(x)$. Also, $\{B_x(t)\}_{x\in\mathbf{Z}^d}$ is a collection of independent $(\mathcal{F}_t)$-Brownian motions on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying the usual right-continuity and completion hypotheses, as will all our filtered probability spaces in this work. We assume that $u_0(x)$ equals 0 except at a finite number of points, $\mathbf{F}$, in $\mathbf{Z}^d$. Pathwise existence and uniqueness holds for solutions of (1.2) by the well-known method of Yamada and Watanabe which we recall below (Lemma 2.1).

If $\gamma = 1$, solutions to (1.2) can be represented in terms of the Feynman-Kac formula. Let $\xi(t)$ be a continuous time random walk on $\mathbf{Z}^d$ with infinitesimal generator Δ and semigroup P_t, which is independent of the Brownian

motions B_x. If $\mathbf{E}_x$ denotes the expectation with respect to ξ, for $\xi(0) = x$, then we have

$$u(t,x) = \mathbf{E}_x \left(u_0(\xi(t)) \exp \left[\int_0^t dB_{\xi(t-s)}(s) - t/2 \right] \right).$$

Since $\exp[\cdot]$ is always strictly positive, and since for each $t > 0$ there is a positive probability that $\xi(t)$ lies in the support of u_0, it follows that $u(t,x) > 0$ for all $t > 0$, $x \in \mathbf{Z}^d$. Gärtner and Molchanov [GM90] have found many fascinating properties of solutions for the case $\gamma = 1$. We also mention in passing that a class of processes called "linear systems" has been studied in the particle systems literature. Such systems are formally similar to solutions of (1.2) with $\gamma = 1$, and Liggett, [Lig85] gives some theorems about the asymptotic die-out of mass as $t \to \infty$.

Next, for $\gamma = 1/2$, the measure $u(t,x)d\mu(x)$, (where μ is the counting measure), is the super-Brownian motion with underlying spatial motion $\xi(t)$. Its total mass satisfies (1.1) and therefore, $u(t,x) = 0$ for all x and large enough t

In light of the above two results it is natural to consider the question of finite time extinction for $1/2 < \gamma < 1$. In this case one can view solutions to (1.2) as interactive super-random walks in which there is a density dependent branching rate of $u(t,x)^{\gamma-1/2}$ at (t,x). Clearly, for some Brownian motion $B(t)$, the total mass $Z(t)$ satisfies

$$dZ(t) = \left(\sum_{x \in \mathbf{Z}^d} u^{2\gamma}(t,x) \right)^{1/2} dB(t). \tag{1.3}$$

Suppose that $H(t) \geq cZ^\gamma(t)$, where $H(t)$ is nonanticipating. It is known that with probability 1, solutions to $dZ = HdB$ die out in finite time. See, for example, Lemma 3.4 of [MP92]. Unfortunately, if $u(t,x)$ is very thinly spread, $[\sum_{x \in \mathbf{Z}^d} u^{2\gamma}(t,x)]^{1/2}$ may be much smaller than $Z^\gamma(t)$. Thus, the coefficient of $dB(t)$ which appears in (1.3) may be much smaller than $Z^\gamma(t)$. This is the main difficulty in proving Theorem 1.

Theorem 1 *Suppose that $1/2 \leq \gamma < 1$, $d \geq 1$, that $u(t,x)$ satisfies (1.2), and that $u_0(x)$ is equal to 0 except on a finite set $\mathbf{F}$. Then, with probability 1, $u(t,x)$ dies out in finite time. That is, there exists an almost surely finite random time $\tau = \tau(\omega)$ such that $u(t,x) = 0$ for all $t \geq \tau$ and $x \in \mathbf{Z}^d$.*

The strategy of our proof is to show that $u(t,x)$ is not thinly spread, and therefore $Z(t)$ satisfies an equation like $dZ = HdB$, where

$$H \geq cZ^\gamma(t) \tag{1.4}$$

for some random number c. It is known that solutions to such equations die out in finite time. Actually, $u(t,x)$ shows a high degree of clumping as x varies. Here, we were guided by known results for superprocesses. Without the clumping, we would not be able to show an inequality such as (1.4).

Finite-time extinction is often useful for establishing the compact support property of solution to continuous parameter parabolic stochastic PDE's. The compact support property states that if the initial data has compact support in $\mathbf{R}$ then the same is true of the solution at any positive time. Often, the compact support property of solutions is proved by showing that finite-time die-out occurs for the parts of the solution corresponding to large values of x. For example, this is done in [MP92] and [DP91]. In fact, Theorem 3.10 of [MP92] proves the continuous analogue of Theorem 1 but we were unable to extend that approach to our lattice systems. At a crucial step in the proof in [MP92], we used Jensen's inequality. To prove Theorem 1 we again use Jensen's inequality, but we also need to know that the mass of $u(t,x)$ tends to cluster at a small number of sites.

Next, we introduce a system of SDE's introduced in [DP98]. Let $M_F(\mathbf{Z}^d)$ be the space of finite measures on $\mathbf{Z}^d$ with the topology of weak convergence. Consider

$$\begin{aligned} dU_t(x) &= \Delta U_t(x)dt + \sqrt{U_t(x)V_t(x)}dB_{1,x}(t), x \in \mathbf{Z}^d \\ dV_t(x) &= \Delta V_t(x)dt + \sqrt{U_t(x)V_t(x)}dB_{2,x}(t), x \in \mathbf{Z}^d \\ U_0, V_0 &\in M_F(\mathbf{Z}^d). \end{aligned} \tag{1.5}$$

Here, $\{B_{i,x}(t)\}_{x\in\mathbf{Z}^d; i=1,2}$ is a collection of independent $\mathcal{F}_t$-Brownian motions ($\mathcal{F}_t$ are as above) and Δ is the discrete Laplacian on $\mathbf{Z}^d$.

Such a pair of processes arise as the large population limit of two interacting branching populations in which the branching rate of each type at $x \in \mathbf{Z}^d$ is proportional to the amount of the other type at x. As each type "catalyzes" the reproduction of the other type, it is called the mutually catalytic branching process. One reason for interest in this system is that it was an extremely simple example of interactive branching for which uniqueness in law was not known. [Myt98] and [DP98] proved weak existence and uniqueness of solutions to (1.5) by means of a self-duality argument proposed by

Mytnik. Uniqueness in law for general systems involving interactive branching rates remains unresolved even for quite smooth rates. The original reason for interest in (1.5) was the qualitative behaviour of its continuum analogues in 2 or more dimensions (the one dimensional case is treated in [DP98]). The singularity of super-Brownian motion (for $d \geq 2$) and the fact that each type solves the heat equation in the absence of the other, suggests that the two types separate and have densities away from their "interface". In [DEF$^+$99] this description is made precise, at least for $d = 2$.

The components $U., V.$ are continuous $M_F(\mathbf{Z^d})$-valued processes a.s., so let P_{U_0,V_0} denote the law of the solution on $C([0,\infty), M_F(\mathbf{Z}^d)^2)$. We set $\langle U, \phi\rangle = \sum_{x\in\mathbf{Z}^d} \phi(x)U(x)$ for bounded ϕ and $U \in M_F(\mathbf{Z}^d)$. The long time behavior of solutions to (1.5) was studied in [DP98]. It is easy to see that $(\langle U_t, 1\rangle, \langle V_t, 1\rangle)$ is a conformal martingale in the first quadrant and hence converges a.s. as $t \to \infty$ to $(\langle U_\infty, 1\rangle, \langle V_\infty, 1\rangle)$, say. [DP98] showed that in the recurrent case ($d \leq 2$), $\langle U_\infty, 1\rangle\langle V_\infty, 1\rangle = 0$ a.s., while in the transient case ($d \geq 3$), $P_{U_0,V_0}(\langle U_\infty, 1\rangle\langle V_\infty, 1\rangle > 0) > 0$. The self-duality then allowed one to use these results to study the long time behaviour from infinite initial conditions. For finite initial conditions the above results lead one to ask:

1. Is there finite-time extinction of one type if $d \leq 2$?

2. How large is $P_{U_0,V_0}(\langle U_\infty, 1\rangle\langle V_\infty, 1\rangle > 0)$ for $d \geq 3$?

The next two results show that, depending on the initial conditions and independent of the dimension, finite time extinction can occur with probability zero or probability very close to one. In particular, this shows that in the transient case, $P_{U_0,V_0}(\langle V_\infty, 1\rangle\langle V_\infty, 1\rangle > 0)$ may be arbitrarily small, depending on (U_0, V_0).

Our first theorem about this system establishes conditions under which finite time die-out does not occur. As usual $U_0P_t(x) = \sum_y U_0(y)p_t(y, x)$, where $p_t(x, y) = P(\xi_t = x|\xi_0 = y)$. If $x = (x_1, \ldots, x_d) \in \mathbf{Z^d}$, then $|x| = \sum_{i=1}^d |x_i|$.

Theorem 2 *Assume that for t large enough (say $t > t_0$),*

$$\liminf_{|x|\to\infty} \frac{U_0P_t(x)}{V_0P_t(x)} = \liminf_{|x|\to\infty} \frac{V_0P_t(x)}{U_0P_t(x)} = 0. \tag{1.6}$$

Then $P_{U_0,V_0}(\langle U_t, 1\rangle\langle V_t, 1\rangle > 0 \quad \forall t > 0) = 1.$

In Proposition 4.1 we give a large class of initial conditions which satisfy (1.6).

Our second result about this case says that under certain conditions, finite time extinction can occur, at least for one of the species.

Theorem 3 *Assume* $\lambda_1 \geq \lambda_2 > 0$*,* $\lambda_0 > 4\lambda_1 - 3\lambda_2$*, and for some* $0 < c_1 \leq c_2$*,*

$$c_1 e^{-\lambda_1 |x|} \leq V_0(x) \leq c_2 e^{-\lambda_2 |x|} \text{ for all } x.$$

For any $\varepsilon > 0$ *and* $t_1 > 0$ $\exists \eta > 0$ *such that if* $U_0(x) \leq \eta e^{-\lambda_0 |x|}$ *then*

$$P_{U_0,V_0}\left(U_t \equiv 0 \quad \forall t \geq t_1\right) \geq 1 - \varepsilon.$$

Note the above conditions are satisfied in particular if $\lambda_0 > \lambda_1 = \lambda_2$.

We will see that to achieve smaller values of ε, we must take smaller values of η. Both of the above results will be stated and proved for more general generators than Δ in Section 4.

If (U, V) solves (1.5), then so does (cU, cV) (with a modified initial condition). We can take $c = K/\eta$ in the above result to see that if λ_i are as above, $M \geq 1$, and $U_0(x) \leq K e^{-\lambda_0 |x|}$, then for any $\varepsilon, t_1 > 0$ there is a c_1 sufficiently large so that the conclusion of Theorem 3 holds whenever

$$c_1 e^{-\lambda_1 |x|} \leq V_0(x) \leq c_1 M e^{-\lambda_2 |x|}.$$

Here is the plan of our article. We will first deal with (1.2). Section 2 contains some lemmas, including uniqueness for (1.2). In Section 3 we prove Theorem 1. In Section 4 we turn to (1.5) and prove more general versions of Theorems 2 and 3.

2 Some Lemmas

In this section, we prove some preliminary facts and lemmas. Our first lemma gives uniqueness for (1.2). This result follows from the method of Yamada and Watanabe (see Theorem V.40.1 of Rogers and Williams [RW87]). Almost the same proof is given below, but we include it for completeness.

Lemma 2.1 *Suppose that* $\sum_{x \in \mathbf{Z}^d} u_0(x) < \infty$*. Then pathwise uniqueness holds for (1.2).*

Proof. Suppose that $u(t,x)$, $v(t,x)$ are 2 solutions of (1.2). An easy application of Fatou's Lemma shows that

$$E(\sum_x u(t,x)+v(t,x)) \le \sum_x u(0,x)+v(0,x) < \infty, \tag{2.1}$$

and in fact with a bit more work one can show equality holds in the above. Note that $||x|^\gamma - |y|^\gamma|^2 \le |x-y|^{2\gamma}$, and since $\gamma > 1/2$, $\int_{0+} u^{-2\gamma} du = \infty$. Therefore, proceeding as in [RW87], section V.40, we find that for each $x \in \mathbf{Z}^d$,

$$\begin{aligned} &|u(t,x)-v(t,x)| \\ &= \int_0^t \mathrm{sgn}(u(s,x)-v(s,x))\,[u^\gamma(s,x)-v^\gamma(s,x)]\,dB_x(s) \\ &\quad + \int_0^t \mathrm{sgn}(u(s,x)-v(s,x))\,[\Delta u(s,x)-\Delta v(s,x)]\,ds. \end{aligned} \tag{2.2}$$

Note that by the definition of Δ,

$$\sum_{x\in\mathbf{Z}^d} |\Delta u(s,x)-\Delta v(s,x)| \le \sum_{x\in\mathbf{Z}^d} (4d)\,|u(s,x)-v(s,x)| \tag{2.3}$$

Taking expectations and summing over $x \in \mathbf{Z}^d$ in (2.2), and using (2.3), we find that

$$\sum_{x\in\mathbf{Z}^d} E\,|u(t,x)-v(t,x)| \le \int_0^t \sum_{x\in\mathbf{Z}^d} (4d)E\,|u(s,x)-v(s,x)|\,ds.$$

Thus, (2.1) and Gronwall's lemma imply that

$$\sum_{x\in\mathbf{Z}^d} E\,|u(t,x)-v(t,x)| = 0$$

and Lemma 2.1 follows. ∎

Standard arguments show weak existence of solutions to (1.2) (e.g., as in section 2 of [DP98]), and just as for finite dimensional SDE (see V.17.1 of [RW87]), this and the above result imply pathwise existence for solutions of (1.2) and the uniqueness of its law on $C([0,\infty), M_F(\mathbf{Z}^d))$.

Let Y be a Poisson random variable with parameter λ, and suppose that H is a non-negative integer. Then, by Stirling's formula,

$$P(Y \ge H) = \sum_{k=0}^{\infty} \frac{\lambda^{(k+H)}}{(k+H)!} e^{-\lambda}$$

$$
\begin{aligned}
&\leq \frac{\lambda^H}{H!}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda} \\
&\leq \frac{C\lambda^H}{\sqrt{H}H^He^{-H}}. \qquad (2.4)
\end{aligned}
$$

The next lemma follows from an easy application of Itô's lemma. See Ch. 5 of [Wal86] for the result in the more delicate continuum setting.

Lemma 2.2 *(1.2) is equivalent to the following system of integral equations.*

$$
\begin{aligned}
u(t,x) &= \sum_{y\in\mathbf{Z}^d} G(t,x-y)u_0(y) \qquad (2.5) \\
&\quad + \int_0^t \sum_{y\in\mathbf{Z}^d} G(t-s,x-y)\cdot u^\gamma(s,y)dB_y(s), \;\; x\in\mathbf{Z}^d,
\end{aligned}
$$

where $G(t,x)$ is the fundamental solution of the discrete heat equation on $\mathbf{Z}^d$.

A standard consequence of pathwise uniqueness is

Lemma 2.3 *(1.2) has the strong Markov property with respect to the σ-fields $\mathcal{F}_t$.*

If $x\in\mathbf{Z}^d$, let x_i denote the i^{th} component of the vector x. Fix an integer $N>0$, and let $D_N=\{x\in\mathbf{Z}^d : |x_i|\leq N \text{ for all } i=1,\dots,d\}$. Note that D_N has at most $(3N)^d$ sites. Let ∂D_N be the boundary of D_N. In other words, let ∂D_N be the set of points in $\mathbf{Z}^d\setminus D_N$ which are nearest neighbors of some point in D_N. Recall that $x,y\in\mathbf{Z}^d$ are nearest neighbors if the Euclidean distance between them is 1. For future use, we denote $\overline{D}_N=D_N\cup\partial D_N$. We reserve the notation $u(t,x)$ for the unique solution of (1.2).

Lemma 2.4 *Let $v_0(x)$ be a non-negative function supported on D_N. Suppose that for $t>0$ and $x\in\mathbf{Z}^d$, $v(t,x)$ satisfies*

$$
\begin{aligned}
dv(t,x) &= \Delta v(t,x)dt + v^\gamma(t,x)dB_x(t) \quad \text{if } x\in D_N \qquad (2.6) \\
v(t,x) &= 0 \quad \text{if } x\in D_N^c \\
v(0,x) &= v_0(x) \quad \text{if } x\in D_N
\end{aligned}
$$

with $v_0(x)\leq u_0(x)$. For $x\in\mathbf{Z}^d\setminus D_N$, let $v(t,x)=0$. Then, with probability 1, for all $(t,x)\in[0,\infty)\times\mathbf{Z}^d$ we have $v(t,x)\leq u(t,x)$.

Proof. The lemma follows from standard comparison arguments. See, for example, [Kot92]. ∎

As in Lemma 2.2, the solution v to (2.6) will satisfy

$$\begin{aligned} v(t,x) &= \sum_{y\in\mathbf{Z}^d} G_N(t,x-y)v_0(y) \\ &\quad + \int_0^t \sum_{y\in\mathbf{Z}^d} G_N(t-s,x-y)\cdot v^\gamma(s,y)dB_y(s),\ x\in D_N, \end{aligned} \tag{2.7}$$

where $G_N(t,x)$ is the fundamental solution of the discrete heat equation on $\mathbf{Z}^d$ with 0 boundary conditions on ∂D_N.

We will at times use the following consequence of Jensen's inequality. Let $M>0$, suppose that $p>1$, and that $a_1,\ldots,a_M$ are non-negative real numbers. Then,

$$\begin{aligned} \sum_{k=1}^M a_k^p &= M\left(\sum_{k=1}^M a_k^p \frac{1}{M}\right) \\ &\geq M\left(\sum_{k=1}^M a_k \frac{1}{M}\right)^p \\ &= M^{1-p}\left(\sum_{k=1}^M a_k\right)^p. \end{aligned} \tag{2.8}$$

For the following lemma, let $\#S$ denote the cardinality of the set S.

Lemma 2.5 *Suppose that $v(t,x)$ satisfies (2.6), and let*

$$V(t) = \sum_{x\in D_N} v(t,x).$$

Let $\partial^- D_N$ be the points in D_N which are nearest neighbors to ∂D_N. Then there exists a Brownian motion $B(t)$ and a predictable functional $H(t) \geq (3N)^{-(2\gamma-1)d/2}V^\gamma(t)$ such that $V(t)$ satisfies the following stochastic differential equation:

$$\begin{aligned} dV(t) &= -\sum_{x\in\partial^- D_N} v(t,x)[2d - \#(\mathcal{N}(x)\cap D_N)]dt \\ &\qquad + H(t)dB(t) \\ V(0) &= \sum_{x\in D_N} v(0,x) \end{aligned} \tag{2.9}$$

Proof. Summing over $x \in D_N$, combining the Brownian motions and using (2.8) with $p = 2\gamma$, we get the lemma. ∎

The following lemma is a special case of Lemma 3.4 of [MP92].

Lemma 2.6 *Let $A > 0$, $\gamma \in (1/2, 1)$, and let $Z(t)$ satisfy*

$$\begin{aligned} dZ(t) &= AZ^\gamma(t)dB(t) \\ Z(0) &= z_0 > 0. \end{aligned}$$

Let τ be the first time t that $Z(t) = 0$, and let $\tau = \infty$ if $Z(t)$ never reaches 0. There is a constant $C(\gamma)$ depending only on γ such that

$$P(Z(t) > 0) = P(\tau > t) \leq C(\gamma)A^{-2}z_0^{2-2\gamma}t^{-1}.$$

Lemma 2.6 has the following simple corollary.

Lemma 2.7 *Let X_t, $t \leq T$, be a continuous non-negative supermartingale with martingale part*

$$\int_0^t H_s dB_s$$

for some Brownian motion B_t and for some predictable process H_t. If $L \geq 1$ and $A > 0$, then

$$P\left(X_T > 0, \int_0^T 1(H_t \geq AX_t^\gamma)dt \geq T/L\right) \leq C(\gamma, L)A^{-2}X_0^{2-2\gamma}T^{-1}.$$

Proof. Let $\sigma(t) = \int_0^{t\wedge T_X} H_u^2 A^{-2} X_u^{-2\gamma} du$, where $T_X = \inf\{t : X_t = 0\}$, and define $\tau(s) = \inf\{t : \sigma(t) > s\}$, if $s < \sigma(T_X)$; and $\tau(s) = T_X$, if $s \geq \sigma(T_X)$. Then $Z_t = X_{\tau(t)}$ is a right continuous non-negative supermartingale with continuous martingale part given by $\int_0^{t\wedge T_Z} AZ_s^\gamma d\tilde{B}_s$ for some Brownian motion $\tilde{B}$. Z may only have negative jumps. Let Y_t be the unique solution of

$$Y_t = X_0 + \int_0^t AY_s^\gamma d\tilde{B}_s.$$

A well-known comparison theorem (Theorem V.43.1 of [RW87] is easily modified to cover this case) shows that $Z_{t-} \leq Y_t$ for all $t \geq 0$ a.s. and so it follows easily that $X_t \leq Y_{\sigma(t)}$ for all $t \geq 0$ a.s. Clearly

$$\int_0^T 1(H_t \geq AX_t^\gamma)dt \geq T/L$$

and $T < T_X$ imply $\sigma(T) \geq T/L$. Recalling that both X and Y will stick at 0 after they first hit 0, we see that

$$\begin{aligned} P\left(X_T > 0, \int_0^T 1(H_t \geq AX_t^\gamma)dt \geq T/L\right) &\leq P\left(\sigma(T) \geq T/L, Y_{\sigma(T)} > 0\right) \\ &\leq P\left(Y_{T/L} > 0\right), \end{aligned}$$

and Lemma 2.6 completes the proof. ∎

Lemma 2.8 *Suppose that f and g are non-negative functions on $\mathbf{Z}^d$ such that*

$$\begin{aligned} \sum_x f(x) &\leq M, \\ \sum_x g(x)f(x) &\leq K. \end{aligned}$$

Then

$$\sum_x g(x)^{2-2\gamma} f(x) \leq K^{2-2\gamma} M^{2\gamma-1}.$$

Proof. By Hölder's inequality,

$$\begin{aligned} \sum_x g(x)^{2-2\gamma} f(x) &= \sum_x \left[(g(x)f(x))^{2-2\gamma} f(x)^{1-(2-2\gamma)}\right] \\ &\leq \left[\sum_x g(x)f(x)\right]^{2-2\gamma} \left[\sum_x f(x)\right]^{2\gamma-1} \\ &\leq K^{2-2\gamma} M^{(2\gamma-1)}. \quad ∎ \end{aligned}$$

Our final result shows that we can split up solutions at $t = 0$ in an appropriate manner. First we observe that since $2\gamma > 1$, if $a, b > 0$ then

$$(a+b)^{2\gamma} \geq a^{2\gamma} + b^{2\gamma}.$$

Lemma 2.9 *Fix $n \geq 0$, and let $u_0 : \mathbf{Z}^d \to [0, \infty)$. For each $1 \leq i \leq n$, let S_i be a finite subset of $\mathbf{Z}^d$, and let $w_{i,0}(\cdot)$ be a non-negative function supported on S_i. Assume that*

$$\sum_{i=1}^n w_{i,0}(x) \leq u_0(x).$$

Then on some filtered probability space, $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, we may define independent $\mathcal{F}_t$-Brownian motions $\{B_{i,x} : 1 \leq i \leq n, x \in \mathbf{Z}^d\}$, independent $\mathcal{F}_t$-Brownian motions $\{B_x : x \in \mathbf{Z}^d\}$ (the 2 collections will not be mutually independent), and non-negative $(\mathcal{F}_t)$-predictable continuous functions $u(\cdot, x)$ and $w_i(\cdot, x)$, $H_i(\cdot, x)$ $(i \leq n, x \in \mathbf{Z}^d)$ such that the following holds. First, the $w_i(t, x)$ satisfy

$$\begin{aligned} dw_i(t,x) &= \Delta w_i(t,x)dt + H_i(t,x)dB_{i,x}(t) \quad \text{if } x \in S_i \\ w_i(t,x) &= 0 \quad \text{if } x \notin S_i \\ w_i(0,x) &= w_{i,0}(x) \end{aligned} \tag{2.10}$$

and $u(t, x)$ satisfies (1.2). Secondly,

$$H_i(t,x) \geq w_i(t,x)^\gamma.$$

Finally, with probability 1, for all $t \geq 0$, $x \in \mathbf{Z}^d$, we have

$$\sum_{i=1}^n w_i(t,x) \leq u(t,x).$$

Proof. We give a brief outline of the proof, leaving the details to the reader. Observe that if the continuous function $h_i : \mathbf{R}^n \to \mathbf{R}_+$ is defined by

$$h_i(w_1, \ldots, w_n) = \frac{w_i^\gamma \left(\sum_{k=1}^n w_k\right)^\gamma}{\left(\sum_{k=1}^n w_k^{2\gamma}\right)^{1/2}} \mathbf{1}\left(\sum_{k=1}^n w_k > 0\right), \tag{2.11}$$

then

$$h_i(w_1, \ldots, w_n) \geq w_i^\gamma$$

and

$$\sum_{i=1}^n h_i^2(w_1, \ldots, w_n) = \left(\sum_{i=1}^n w_i\right)^{2\gamma}.$$

Let $h_i^{(m)}$ be a sequence of Lipschitz functions on $\mathbf{R}^n$, which converge to h_i uniformly. More specifically, we define a function $\phi_m : \mathbf{R}_+ \to \mathbf{R}$ as follows. Let $\phi_m(w)$ agree with w^γ on $[1/m, \infty)$ and at 0, and define $\phi_m(w)$ by linear interpolation on $(0, 1/m)$. Define $h_i^{(m)}$ as h_i but with $\phi_m(w_i)\phi_m(\sum_{k=1}^n w_k)$ in the

numerator. It follows that $\sum_{i=1}^n h_i^{(m)^2}(w_1, \ldots, w_n)$ converges to $(\sum_{i=1}^n w_i)^{2\gamma}$ uniformly on compacts. Let $w_i^{(m)}(t,x)$ be the unique solution of

$$\begin{aligned}
dw_i^{(m)}(t,x) &= \Delta w_i^{(m)}(t,x)dt + h_i^{(m)}(w_1^{(m)}(t,x), \ldots, w_n^{(m)}(t,x))dB_{i,x}(t) \\
&\qquad \text{if } x \in S_i \\
w_i^{(m)}(t,x) &= 0 \quad \text{if } x \notin S_i \\
w_i^{(m)}(0,x) &= w_{i,0}(x).
\end{aligned}$$

Let $z_i^{(m)}(t,x)$ be the unique solution of

$$\begin{aligned}
dz_i^{(m)}(t,x) &= \Delta z_i^{(m)}(t,x)dt + h_i^{(m)}(z_1^{(m)}(t,x), \ldots, z_n^{(m)}(t,x))dB_{i,x}(t) \\
z_i^{(m)}(0,x) &= \bar{w}_{i,0}(x),
\end{aligned}$$

where $\bar{w}_{i,0}(x) \geq w_{i,0}(x)$ are chosen such that

$$\sum_{i=1}^m \bar{w}_{i,0}(x) = u_0(x).$$

Then, standard comparison theorems show that with probability 1,

$$0 \leq w_i^{(m)}(t,x) \leq z_i^{(m)}(t,x) \text{ for all } t, x.$$

Furthermore, we can take a subsequence m_k such that $(w_i^{(m_k)}(t,x), z_i^{(m_k)}(t,x))$ converges in distribution to $(w_i(t,x), z_i(t,x))$, where (w_i) satisfy (2.10) with $H_i(t,x) = h_i(w_1(t,x), \ldots, w_n(t,x))$ and $\sum_{i=1}^n z_i(t,x)$ satisfies (1.2). Clearly with probability 1,

$$0 \leq w_i(t,x) \leq z_i(t,x) \text{ for all } t, x, i. \tag{2.12}$$

This implies Lemma 2.9. ∎

3 Proof of Theorem 1

Our proof depends heavily on the decomposition in Lemma 2.9. We emphasize that this decomposition is a weak existence theorem, so we are always dealing with a changing set of Brownian motions and solutions.

First, we set up some notation which we will use in the proof. Fix $\varepsilon > 0$. Let $\mathcal{K}$ be the event that for some $T > 0$, $u(T,x) = 0$ for all $x \in \mathbf{Z}^d$. For

$T \geq 0$, let $\mathcal{K}(T)$ be the event that $u(T,x) = 0$ for all $x \in \mathbf{Z}^d$. Note that as $T \uparrow \infty$, the events $\mathcal{K}(T)$ increase to $\mathcal{K}$. In fact, we show that there exists a time t_∞ such that

$$P\left(\mathcal{K}(t_\infty)^c\right) < \varepsilon. \tag{3.1}$$

Then (3.1) implies Theorem 1. From now on, we fix $\varepsilon > 0$ and concentrate on proving (3.1).

Let $\ell \in \mathbf{N}$ be so large that

$$d \cdot \gamma^{\ell-1} < \frac{1}{8}, \tag{3.2}$$

and then choose $0 < \delta$ small enough so that

$$\delta < \frac{1}{2}\gamma^{\ell-1}(1-\gamma). \tag{3.3}$$

We will specify an integer $n_0 > 0$ later. Let

$$N_n = 2^n.$$

We define $0 = t_{n_0} < t_{n_0+1} < \ldots$ inductively as follows. Let $\bar{m}$ be the smallest integer m such that

$$m\ell \geq 2^{n_0/2}$$

and let

$$t_{n_0+1} = \bar{m}\ell.$$

If $n > n_0$ and if we are given t_n, we let

$$t_{n+1} = t_n + 2^{-n/2}.$$

Clearly, there exists a finite accumulation point

$$t_\infty = \lim_{n\to\infty} t_n < \infty.$$

Fix $K > 0$ and let $\mathcal{A}_0 = \mathcal{A}_0(K)$ be the event that

$$\sup_{t\geq 0} \sum_{x\in\mathbf{Z}^d} u(t,x) \leq K.$$

Let $\tilde{M}_t = \sum_{x\in\mathbf{Z}^d} u(t,x)$. Note that the integrability in (2.1) easily gives that $\tilde{M}_t$ is a continuous non-negative martingale. Let $\tilde{\tau} = \tilde{\tau}(T,K)$ be the first

time $t \le T$ that $\tilde{M}_t \ge K$. If there is no such time, let $\tilde{\tau} = T$. Using the optional sampling theorem and Markov's inequality, we get

$$\begin{aligned} P\left(\mathcal{A}_0(K)^c\right) &= \lim_{T\to\infty} P\left(\tilde{M}_{\tilde{\tau}(T,K)} \ge K\right) \\ &\le \frac{\tilde{M}_0}{K}. \end{aligned} \tag{3.4}$$

For $n \ge n_0$, $t_n \le t \le t_{n+1}$, $x \in \mathbf{Z}^d$, we will inductively define a sequence of random functions $v_n(t,x) \le u(t,x)$ as follows. Since the definition of $v_n(t,x)$ for $n > n_0$ is the simplest, we start with that case. Suppose that we have defined $v_{n-1}(t,x) \le u(t,x)$ for $t_{n-1} \le t \le t_n$. Let

$$v_n(t_n,x) = u(t_n,x) - v_{n-1}(t_n,x). \tag{3.5}$$

For $t_n < t \le t_{n+1}$, let $v_n(t,x)$ satisfy equation (2.6) (of Lemma 2.4), with $N = N_n$, so that $v_n(t,x) \le u(t,x)$ by Lemma 2.4.

Now we give the more complicated definition of $v_{n_0}(t,x)$. For $1 \le m \le \bar{m}$, we call the time intervals $[(m-1)(\ell), m(\ell))$ stages, and we call the subintervals $[k,k+1) \subset [(m-1)(\ell), m(\ell))$ substages. In order to define $v_{n_0}(t,x)$, we first define a collection of functions $w_{k,z}(t,x)$ for $k\ell \le t \le (k+1)\ell$, $0 \le k < \bar{m}$, $z \in D_{N_{n_0}}$, satisfying

$$\sum_{z\in D_{N_{n_0}}} w_{k,z}(t,x) \le u(t,x), \quad k\ell \le t \le (k+1)\ell. \tag{3.6}$$

We let $x + D_k$ denote the set $\{x+y : y \in D_k\}$. Let

$$\bar{N}_n = 2^{\delta n}.$$

Assume that either $k=0$ or that $w_{k-1,z}$ has already been defined for $z \in D_{N_{n_0}}$ and satisfies (3.6). Let

$$w_{0,z}(0,x) = u(0,z)\mathbf{1}(x=z),$$

and let

$$w_{k,z}(k\ell,x) = \sum_{y\in D_{N_{n_0}}} w_{k-1,y}(k\ell,z)\mathbf{1}(x=z).$$

Therefore (3.6) holds for $t = k\ell$. In either case, using Lemma 2.9 with $S_z = z + D_{\bar{N}_{n_0}}$, $z \in D_{N_{n_0}}$, we let $w_{k,z}(t,x)$ satisfy the following equation for $k\ell \le t \le (k+1)\ell$.

$$\begin{aligned} dw_{k,z}(t,x) &= \Delta w_{k,z}(t,x)dt + H_z(t,x)dB_{z,x}(t) \quad x \in z + D_{\bar{N}_{n_0}} \\ w_{k,z}(t,x) &= 0, \qquad x \notin z + D_{\bar{N}_{n_0}}, \end{aligned}$$

where $H_z(t,x)$ is as in Lemma 2.9, and $\{B_{z,x}(t)\}_{z,x}$ are independent Brownian motions. Lemma 2.9 now implies (3.6) and our inductive construction is complete.

Finally, for $0 \le t \le t_{n_0+1}$, we define $v_{n_0}(t,x)$ as follows. Let $D^0_{N_{n_0}}$ be the set of those points $z \in D_{N_{n_0}}$ such that

$$z + D_{\bar{N}_{n_0}} \subset D_{N_{n_0}}.$$

The reason for defining $D^0_{N_{n_0}}$ is that we do not want to include any points z for which $w_{k,z}(t,x)$ has any support outside of $D_{N_{n_0}}$. If $x \notin D^0_{N_{n_0}}$, let $v_{n_0}(t,x) = 0$. If $x \in D^0_{N_{n_0}}$ and $t \le \bar{m}\ell = t_{n_0+1}$, choose $0 \le k < \bar{m}$ such that $k\ell \le t \le (k+1)\ell$ and let

$$v_{n_0}(t,x) = \sum_{z \in D^0_{N_{n_0}}} w_{k,z}(t,x) \le u(t,x).$$

Extend v_n, $w_{k,z}$ to be identically 0 outside their initial domains of definition and let $\mathcal{F}_t$ be the right-continuous filtration generated by the processes v_n, $w_{k,z}$, u, and $B_{z,x}$ up to time t as n,k,z,x range through their respective domains. Now we label the mass that has "leaked out". For $n \ge n_0$, let

$$M_n = \sum_{x \in \mathbf{Z}^d} [u(t_{n+1},x) - v_n(t_{n+1},x)].$$

We will often work with the following sets.

Definition 3.1 *For $n \ge n_0$ let $\mathcal{A}_{1,n}$ be the event that*

$$v_n(t_{n+1},x) = 0 \textit{ for all } x \in \mathbf{Z}^d,$$

and let $\mathcal{A}_{2,n}$ be the event that

$$M_n \le 2^{-2^{\delta n}}.$$

Define $\mathcal{A}_{i,n_0-1}$ to be the entire space for $i = 1$ or 2.

Our definitions imply that

$$\mathcal{K}(t_\infty) \supset \mathcal{A}_0(K) \cap \left[\bigcap_{n=n_0}^{\infty} \mathcal{A}_{1,n}\right] \cap \left[\bigcap_{n=n_0}^{\infty} \mathcal{A}_{2,n}\right].$$

The following lemma plays an essential role in the proof of Theorem 1.

Lemma 3.1 *If n_0 is large enough and $n \geq n_0$, then*

$$P\left(\mathcal{A}_{2,n}^c \cap \mathcal{A}_0(K) \cap \mathcal{A}_{1,n-1}\right) \leq 2^{-2^{\delta n}},$$

and in particular, for n_0 large enough,

$$\begin{aligned} P\left(\bigcup_{n=n_0}^{\infty}\left[\mathcal{A}_{2,n}^c \cap \mathcal{A}_0(K) \cap \mathcal{A}_{1,n-1}\right]\right) &\leq \sum_{n=n_0}^{\infty} 2^{-2^{\delta n}} \\ &\leq \frac{\varepsilon}{4}. \end{aligned} \tag{3.7}$$

Proof. To begin the proof of Lemma 3.1, we consider the case $n > n_0$. The key to the proof is the observation that on the set $\mathcal{A}_{1,n-1}$ we have $v_n(t_n, \cdot) = u(t_n, \cdot)$. This follows from the definitions of v_n and the event $\mathcal{A}_{1,n-1}$. Let ξ_t^x be our original continuous time random walk, started from x, and let τ_n^x be the first time t that $\xi_t^x \notin D_{N_n}$. The integral equations (2.5) and (2.7) imply that on $\mathcal{A}_{1,n-1} \in \mathcal{F}_{t_n}$,

$$E\left(\sum_x u(t_{n+1}, x) - v_n(t_{n+1}, x)\middle|\mathcal{F}_{t_n}\right) = \sum_x u(t_n, x) P(\tau_n^x < 2^{-n/2}). \tag{3.8}$$

Now use (2.5) again to see that

$$\begin{aligned} &E\left(M_n \mathbf{1}(\mathcal{A}_0(K) \cap \mathcal{A}_{1,n-1})\right) \\ &\leq \mathbf{1}\Big(\sum_x u_0(x) \leq K\Big) E\left(\mathbf{1}(\mathcal{A}_{1,n-1}) E\left(\sum_x u(t_{n+1}, x) - v_n(t_{n+1}, x)\middle|\mathcal{F}_{t_n}\right)\right) \\ &\leq \mathbf{1}\Big(\sum_x u_0(x) \leq K\Big) E\left(\sum_x u(t_n, x) P(\tau_n^x < 2^{-n/2})\right) \\ &\leq \mathbf{1}\Big(\sum_x u_0(x) \leq K\Big) \sum_x u_0(x) P(\tau_n^x < t_n + 2^{-n/2} = t_{n+1}) \\ &\leq K \sup_{x \in \mathbf{F}} P(\tau_n^x < t_{n+1}). \end{aligned}$$

Denote

$$\|\mathbf{F}\| = \max_{x \in \mathbf{F}} |x|.$$

Let S_t be the number of steps that ξ_s has taken for $s \leq t$. Of course, since the steps are of size 1, we have that

$$P\left(\tau_n^x \leq t_{n+1}\right) \leq P\left(S_{t_{n+1}} > 2^n - \|\mathbf{F}\|\right).$$

Recall the definition of M_n. Also, from the definition of t_{n_0} and t_n, we see that for $n_0 \geq n(\ell)$,

$$t_{n+1} \leq t_{n_0} + \sum_{m=n_0}^{\infty} 2^{-m/2} \leq 2^{1+n_0/2}.$$

Using (2.4), we conclude that

$$\begin{aligned} E\left[M_n \mathbf{1}(\mathcal{A}_0(K) \cap \mathcal{A}_{1,n-1})\right] &\leq KP\left(S_{t_{n+1}} > 2^n - \|\mathbf{F}\|\right) \\ &\leq \frac{CK(2dt_{n+1})^{2^n - \|\mathbf{F}\|}}{\sqrt{2^n - \|\mathbf{F}\|}\,(2^n - \|\mathbf{F}\|)^{2^n - \|\mathbf{F}\|}\, e^{-2^n + \|\mathbf{F}\|}} \\ &\leq 8^{-2^n} \end{aligned} \tag{3.9}$$

if n is large enough, and thus if n_0 is large enough. Using Markov's inequality, we have

$$P\left(\{M_n > 2^{-2^n}\} \cap \mathcal{A}_0(K) \cap \mathcal{A}_{1,n-1}\right) \leq 4^{-2^n}.$$

This proves Lemma 3.1 for $n > n_0$ because $\delta < 1$.

Next, we turn to the case $n = n_0$. Note that M_{n_0} consists of 2 kinds of mass. Let $M_{n_0}^{'}$ refer to the first kind of mass, which escapes from each of the small cubes $x + D_{\bar{N}_{n_0}}$. Let $M_{n_0}^{''}$ refer to the second kind of mass, which escapes from the large cube $D_{N_{n_0}}$ or becomes part of the functions $w_{k,z}(t,x)$, for those $z \in D_{N_{n_0}}^0$. To be precise, let

$$M_{n_0}^{'} = \sum_{x \in \mathbf{Z}^d} \left[u(t_{n_0+1}, x) - \sum_{z \in D_{N_{n_0}}} w_{\bar{m}-1,z}(t_{n_0+1}, x) \right]$$

and let

$$M_{n_0}^{''} = \sum_{x \in \mathbf{Z}^d} \left[\sum_{z \in D_{N_{n_0}} - D_{N_{n_0}}^0} w_{\bar{m}-1,z}(t_{n_0+1}, x) \right].$$

Clearly $M_{n_0} = M_{n_0}^{'} + M_{n_0}^{''}$.

First we deal with $M_{n_0}^{'}$. Let $\bar{\tau}$ be the first time $t < \ell$ that $\xi_t \notin D_{\bar{N}_{n_0}}$ where $\xi_0 = 0$. If there is no such time, let $t = \ell$. Using the analogue of (2.7) for the $w_{k,z}$, as in (3.8), we get

$$E\left(\sum_x \left[u(t_{n_0+1}, x) - \sum_z w_{\bar{m}-1,z}(t_{n_0+1}, x) \right] \middle| \mathcal{F}_{(\bar{m}-1)\ell} \right)$$

$$
\begin{aligned}
&= \sum_x \left[u((\bar m-1)\ell,x) - \sum_z w_{\bar m-1,z}((\bar m-1)\ell,x) P(\bar\tau \geq \ell) \right] \\
&= \sum_x \left[u((\bar m-1)\ell,x) - \sum_z w_{\bar m-1,z}((\bar m-1)\ell,x) \right] \\
&\qquad + \sum_{x,z} w_{\bar m-1,z}((\bar m-1)\ell,x) P(\bar\tau < \ell) \\
&\leq \sum_x \left[u((\bar m-1)\ell,x) - \sum_z w_{\bar m-2,z}((\bar m-1)\ell,x) \right] \\
&\qquad + \sum_x u((\bar m-1)\ell,x) P(\bar\tau < \ell).
\end{aligned}
$$

In the last line we have used (3.6) and the fact that at times $k\ell$, the redistribution of the mass among the $w_{k,z}$'s preserves the total mass of $\sum_z w_{k,z}$. Therefore

$$
\begin{aligned}
&E\left(\sum_x \left[u(t_{n_0+1},x) - \sum_z w_{\bar m-1,z}(t_{n_0+1},x) \right] \mathbf{1}(\mathcal{A}_0(K)) \right) \\
&\quad \leq E\left(\sum_x \left[u((\bar m-1)\ell,x) - \sum_z w_{\bar m-2,z}((\bar m-1)\ell,x) \right] \mathbf{1}(\mathcal{A}_0(K)) \right) \\
&\qquad + E\left(\sum_x u((\bar m-1)\ell,x) \mathbf{1}(\sum_x \bar u((\bar m-1)\ell,x) \leq K) \right) P(\bar\tau < \ell).
\end{aligned}
$$

The last term is at most $KP(\bar\tau < \ell)$. Now iterate the above $\bar m$ times, noting that $\sum_x u(0,x) = \sum_x \sum_z w_{0,z}(0,x)$, and argue as in (3.9) using (2.4) to get (for n_0 large again)

$$
\begin{aligned}
E\left[M'_{n_0} \mathbf{1}(\mathcal{A}_0(K)) \right] &\leq K\bar m P(\bar\tau < \ell) \\
&\leq K 2^{n_0/2} P(S_\ell \geq 2^{\delta n_0}) \\
&\leq CK2^{n_0/2} \frac{(2d\ell)^{2^{\delta n_0}}}{\sqrt{2^{\delta n_0}}\,(2^{\delta n_0})^{2^{\delta n_0}}\, e^{-2^{\delta n_0}}} \\
&\leq 8^{-2^{\delta n_0}},
\end{aligned}
$$

if n_0 is large enough. Again using Markov's inequality, we have

$$
P\left(M'_{n_0} > \frac{1}{2} 2^{-2^{\delta n_0}}, \mathcal{A}_0(K) \right) \leq \frac{1}{2} 2^{-2^{\delta n_0}}. \tag{3.10}
$$

Now we consider M''_{n_0}. On the last interval $[(\bar{m}-1)\ell, \bar{m}\ell]$,

$$\sum_{z \in D_{N_{n_0}} \backslash D^0_{N_{n_0}}} \sum_x w_{\bar{m}-1,z}(t,x)$$

is a supermartingale and so

$$\begin{aligned}
&E\left(M''_{n_0}\mathbf{1}(\mathcal{A}_0(K))\right) \\
&\leq \; E\left(\sum_{z\in D_{N_{n_0}}-D^0_{N_{n_0}}} \sum_x w_{\bar{m}-1,z}((\bar{m}-1)\ell,x)\mathbf{1}(\langle u_0,\mathbf{1}\rangle \leq K)\right) \\
&= \; E\left(\sum_{z\in D_{N_{n_0}}-D^0_{N_{n_0}}} \sum_x \sum_{y\in D_{N_{n_0}}} w_{\bar{m}-2,y}((\bar{m}-1)\ell,x)\mathbf{1}(x=z)\right) \\
&\qquad \cdot\mathbf{1}(\langle u_0,\mathbf{1}\rangle \leq K) \\
&= \; E\left(\sum_{x\in D_{N_{n_0}}-D^0_{N_{n_0}}} \sum_{y\in D_{N_{n_0}}} w_{\bar{m}-2,y}((\bar{m}-1)\ell,x)\right)\mathbf{1}(\langle u_0,\mathbf{1}\rangle \leq K).
\end{aligned}$$

Use (3.6) to bound the above by (n_0 large enough)

$$\begin{aligned}
&E\left(\sum_{x\in D_{N_{n_0}}-D^0_{N_{n_0}}} u((\bar{m}-1)\ell,x)\right)\mathbf{1}(\langle u_0,\mathbf{1}\rangle \leq K) \\
&\leq \; K\sup_{x\in\mathbf{F}} P(\xi_x \text{ exits } D^0_{N_{n_0}} \text{ before time } (\bar{m}-1)\ell) \\
&\leq \; KP(S_{2^{n_0/2}} > 2^{n_0-1}).
\end{aligned}$$

Another application of (2.4) (as in (3.9)) shows that for n_0 large enough the above is at most $8^{-2^{n_0}}$, and therefore, by Markov's inequality,

$$P\left(M''_{n_0} > \frac{1}{2}2^{-2^{\delta n_0}}, \mathcal{A}_0(K)\right) \leq \frac{1}{2}2^{-2^{\delta n_0}}. \tag{3.11}$$

Putting together (3.10) and (3.11), we obtain

$$\begin{aligned}
P\left(M_{n_0} > 2^{-2^{\delta n_0}}, \mathcal{A}_0(K)\right) &\leq \; P\left(M'_{n_0} > \frac{1}{2}2^{-2^{\delta n_0}}, \mathcal{A}_0(K)\right) \\
&\quad +P\left(M''_{n_0} > \frac{1}{2}2^{-2^{\delta n_0}}, \mathcal{A}_0(K)\right) \\
&\leq \; 2^{-2^{\delta n_0}}.
\end{aligned}$$

This proves Lemma 3.1. ∎

Our next goal is to estimate the probability of $\mathcal{A}_{1,n}^c$. Let

$$V_n(t) = \sum_{x \in D_{N_n}} v_n(t_n + t, x).$$

Lemma 3.2 *If $n_0 > n(\delta, \varepsilon)$, then for $n > n_0$,*

$$P\left(\mathcal{A}_{1,n}^c \cap \mathcal{A}_{2,n-1}\right) \leq 2^{n_0 - n}\frac{\varepsilon}{4}.$$

Proof. Our argument uses Lemma 2.7. We need to bound

$$P\left(\mathcal{A}_{1,n}^c \cap \mathcal{A}_{2,n-1}\right) = P\left(V_n(2^{-n/2}) > 0, \mathcal{A}_{2,n-1}\right).$$

By Lemma 2.5, on the event $\mathcal{A}_{2,n-1}$, we can write

$$\begin{aligned} dV_n(t) &\leq H(t)dB(t), \qquad 0 < t < t_{n+1} - t_n = 2^{-n/2} \\ V_n(0) &\leq 2^{-2^{\delta(n-1)}}, \end{aligned}$$

where

$$\begin{aligned} H(t) &\geq (3N_n)^{-(2\gamma-1)d/2} V_n(t)^\gamma \\ &= C2^{-nd(\gamma-1/2)} V_n(t)^\gamma. \end{aligned}$$

Then, by Lemma 2.7, we have

$$P\left(V_n(2^{-n/2}) > 0, \mathcal{A}_{2,n-1}\right) \leq C2^{nd(2\gamma-1)} 2^{-2^{\delta(n-1)}(2-2\gamma)} 2^{n/2}.$$

This proves Lemma 3.2, if n_0 is large enough. ∎

Next, we treat the more complicated case of $n = n_0$.

Lemma 3.3 *If $n_0 \geq n(\varepsilon, K)$, then*

$$P\left(\mathcal{A}_{1,n_0}^c \cap \mathcal{A}_0(K)\right) \leq \frac{\varepsilon}{4}.$$

To prove this we will deal with the stages (of length ℓ) and the substages (of length 1) which we defined earlier. We first show that for at least half of the stages, at the end of the $\ell - 1$ substages, there are only a small number of sites $z \in D_{N_{n_0}}$ such that $w_{k,z}$ is still alive. (Recall that we say a function

is alive if it is not identically 0). To state this key lemma precisely, for $0 \le k \le \bar{m}$, we let

$$W_{k,z}(t) = \sum_{x \in z + D_{\bar{N}_{n_0}}} w_{k,z}(t,x), \quad k\ell \le t \le (k+1)\ell,$$

and for $0 \le j < \ell$ and k as above, set

$$\mathcal{A}_{3,k}(j) = \left\{ \sum_{z \in D_{N_{n_0}}} \mathbf{1}(W_{k,z}(k\ell + j) > 0) \le 3^{d\gamma^j} 2^{n_0 d\gamma^j} \right\}.$$

Then we have the following.

Lemma 3.4 *If $n_0 \ge n(\ell, K, \gamma, \varepsilon)$ then*

$$P\left(\left\{ \sum_{k=0}^{\bar{m}-1} \mathbf{1}(\mathcal{A}_{3,k}(\ell-1)^c) \ge \bar{m}/2 \right\} \cap \mathcal{A}_0(K) \right) < \varepsilon/8.$$

Assume for the moment that Lemma 3.4 holds and let us give the
Proof of Lemma 3.3. Note that, as in Lemma 2.5, $W_{k,z}$ is a non-negative supermartingale with martingale part $H_{k,z}dB$, where

$$\begin{aligned} H_{k,z}(t) &\ge \sum_{x \in z + D_{\bar{N}_{n_0}}} w_{k,z}(t,x)^\gamma \\ &\ge \left(\sum_{x \in z + D_{\bar{N}_{n_0}}} w_{k,z}(t,x)^{2\gamma} \right)^{1/2} \\ &\ge \left(3^d 2^{\delta n_0 d} \left(\frac{W_{k,z}(t)}{3^d 2^{\delta n_0 d}} \right)^{2\gamma} \right)^{1/2} \qquad \text{(by (2.8))} \\ &= C 2^{-\delta n_0 d(\gamma - 1/2)} W_{k,z}(t)^\gamma. \end{aligned} \tag{3.12}$$

Lemma 2.5 also shows that $V_{n_0}(t) = \sum_{z \in D_{N_{n_0}}} W_{k,z}(t)$ (for $k\ell \le t < (k+1)\ell$) is a continuous supermartingale with martingale part $H_t dB_t$, where by (3.12) and Jensen's inequality, for $k\ell \le t < (k+1)\ell$

$$\begin{aligned} H_t^2 = \sum_z H_{k,z}(t)^2 &\ge C 2^{-\delta n_0 d(2\gamma-1)} \sum_z W_{k,z}(t)^{2\gamma} \\ &\ge C 2^{-\delta n_0 d(2\gamma-1)} \left[\sum_z \mathbf{1}(W_{k,z}(t) > 0) \right]^{1-2\gamma} V_{n_0}(t)^{2\gamma}. \end{aligned}$$

Therefore on $\mathcal{A}_{3,k}(\ell-1)$ and for $t \in [k\ell+\ell-1,(k+1)\ell]$, we have (note that $3^{d\gamma^{\ell-1}} \le c$)

$$\begin{aligned} H_t &\ge C2^{-\delta n_0 d(\gamma-1/2)} 2^{n_0 d\gamma^{\ell-1}(1/2-\gamma)} V_{n_0}(t)^\gamma \\ &\equiv AV_{n_0}(t)^\gamma. \end{aligned}$$

Thus we may apply Lemma 2.7 along with our choices of ℓ and δ (recall (3.2) and (3.3)) to conclude that

$$\begin{aligned} &P\left(\{V_{n_0}(t_{n_0+1})>0\} \cap \mathcal{A}_0(K) \cap \left\{\sum_{k=0}^{\bar m-1} \mathbf{1}(\mathcal{A}_{3,k}(\ell-1)) > \frac{\bar m}{2}\right\}\right) \\ &\le P\Big(\{V_{n_0}(t_{n_0+1})>0\} \cap \mathcal{A}_0(K) \\ &\quad \cap \left\{\int_0^{t_{n_0+1}} \mathbf{1}(H_t \ge AV_{n_0}(t)^\gamma)dt \ge \frac{\bar m}{2} = \frac{t_{n_0+1}}{2\ell}\right\}\Big) \\ &\le CA^{-2}K^{2-2\gamma}2^{-n_0/2} \\ &= C2^{n_0 d(2\gamma-1)(\delta+\gamma^{\ell-1})-n_0/2}K^{2-2\gamma} \\ &\le C2^{n_0(d\frac{3}{2}\gamma^{\ell-1}-\frac{1}{2})}K^{2-2\gamma} \\ &\le C2^{-n_0/4}K^{2-2\gamma} < \varepsilon/8 \end{aligned}$$

for $n_0 \ge n(\varepsilon, K)$. This together with Lemma 3.4 completes the proof of Lemma 3.3. ∎

We now turn to the

Proof of Lemma 3.4. Fix $0 \le k < \bar m$. For $0 \le j < \ell$ let $\eta_{j,k} = \sum_z \mathbf{1}(W_{k,z}(k\ell+j)>0)$ and

$$\mathcal{A}_{0,k}(j,K) = \left\{\sup_{t \le k\ell+j} \sum_x u(t,x) \le K\right\} \supset \mathcal{A}_0(K).$$

By (3.12) we may use Lemma 2.7 to see that for $1 \le j < \ell$,

$$\begin{aligned} &E\left(\eta_{j,k}\mathbf{1}(\mathcal{A}_{0,k}(j,K))\middle|\mathcal{F}_{k\ell+j-1}\right) \\ &\le \sum_z \mathbf{1}(\mathcal{A}_{0,k}(j-1,K)) P\left(W_{k,z}(k\ell+j)>0\middle|\mathcal{F}_{k\ell+j-1}\right) \\ &\le \mathbf{1}(\mathcal{A}_{0,k}(j-1,K)) \sum_z C2^{\delta n_0 d(2\gamma-1)} W_{k,z}(k\ell+j-1)^{2-2\gamma}. \end{aligned}$$

Now note that on $\mathcal{A}_{0,k}(j-1,K)$, we have $\sum_z W_{k,z}(k\ell+j-1) \leq K$ (by (3.6)). We apply Lemma 2.8 with

$$g(z) = W_{k,z}(k\ell + j - 1) \text{ and } f(z) = \mathbf{1}(W_{k.z}(k\ell + j - 1) > 0)$$

to see that on $\mathcal{A}_{3,k}(j-1)$,

$$\begin{aligned} E\left(\eta_{j,k}\mathbf{1}(\mathcal{A}_{0,k}(j,K))\middle|\mathcal{F}_{k\ell+j-1}\right) &\leq C2^{2\delta n_0 d(2\gamma-1)}K^{2-2\gamma}\eta_{j-1,k}^{2\gamma-1} \\ &\leq C2^{n_0 d(\gamma^{j-1}+\delta)(2\gamma-1)}K^{2-2\gamma}. \end{aligned}$$

Markov's inequality implies for $1 \leq j < \ell$,

$$\begin{aligned} &P\left(\mathcal{A}_{3,k}(j)^c \cap \mathcal{A}_{0,k}(j,K) \cap \mathcal{A}_{3,k}(j-1)\middle|\mathcal{F}_{k\ell+j-1}\right) \\ &\leq \frac{C2^{n_0 d(\gamma^{j-1}+\delta)(2\gamma-1)}K^{2-2\gamma}}{3^{d\gamma^j}2^{n_0 d\gamma^j}} \\ &\leq CK^{2-2\gamma}2^{n_0 d(\delta(2\gamma-1)-\gamma^{j-1}(1-\gamma))} \\ &\leq CK^{2-2\gamma}2^{-n_0 d\gamma^\ell(1-\gamma)/2} \equiv p'_{n_0}, \end{aligned}$$

the last by (3.3). Since there are at most $3^d 2^{n_0 d}$ sites z in $D_{N_{n_0}}$, $P\left(\mathcal{A}_{3,k}(0)\right) = 1$ and so from the above we have,

$$\begin{aligned} &P\left(\mathcal{A}_{3,k}(\ell-1)^c \cap \mathcal{A}_{0,k}(\ell-1,K)\middle|\mathcal{F}_{k\ell}\right) \\ &\leq P\left(\bigcup_{j=1}^{\ell-1}\left[\mathcal{A}_{3,k}(j)^c \cap \mathcal{A}_{0,k}(j,K)\right]\middle|\mathcal{F}_{k\ell}\right) \\ &\leq \sum_{j=1}^{\ell-1} P\left(\mathcal{A}_{3,k}(j)^c \cap \mathcal{A}_{0,k}(j,K) \cap (\mathcal{A}_{3,k}(j-1) \cup \mathcal{A}_{0,k}(j-1,K)^c)\middle|\mathcal{F}_{k\ell}\right) \\ &= \sum_{j=1}^{\ell-1} P\left(\mathcal{A}_{3,k}(j)^c \cap \mathcal{A}_{0,k}(j,K) \cap \mathcal{A}_{3,k}(j-1)\middle|\mathcal{F}_{k\ell}\right) \\ &\leq \ell p'_{n_0} \equiv p_{n_0} < \frac{1}{4}, \end{aligned}$$

if $n_0 > n(\ell, K, \gamma)$. Allowing $k < \bar{m}$ to vary, let

$$d_k = \mathbf{1}(\mathcal{A}_{3,k}(\ell-1)^c \cap \mathcal{A}_{0,k}(\ell-1,K)) \in \mathcal{F}_{(k+1)\ell},$$

and $M_n = \sum_{k=1}^{n-1} d_k - E(d_k|\mathcal{F}_{k\ell})$, $n < \bar{m}$. Clearly $(M_n, \mathcal{F}_{n\ell})_{n<\bar{m}}$ is a martingale and so if $n_0 \geq n(\ell, K, \gamma, \varepsilon)$,

$$\begin{aligned}
P(\sum_{k=0}^{\bar{m}-1} d_k \geq \bar{m}/2) &\leq P\left(M_{\bar{m}} \geq \bar{m}\left(\frac{1}{2} - p_{n_0}\right)\right) \\
&\leq E\left(M_{\bar{m}}^2\right) \bar{m}^{-2} \left(\frac{1}{2} - p_{n_0}\right)^{-2} \\
&\leq 16\bar{m}^{-2} E\left(\sum_{k=1}^{\bar{m}-1} d_k^2\right) \\
&\leq 16\bar{m}^{-1} p_{n_0} < \frac{\varepsilon}{8}.
\end{aligned}$$

The probability we have to bound is no bigger than that on the left hand side of the above equation and so the proof is complete. ∎

Now we can complete the

Proof of Theorem 1. Recall that

$$\mathcal{K}(t_\infty) \supset \mathcal{A}_0(K) \cap \left[\bigcap_{n=n_0}^{\infty} \mathcal{A}_{1,n}\right] \cap \left[\bigcap_{n=n_0}^{\infty} \mathcal{A}_{2,n}\right].$$

Using (3.4), choose K so large that

$$P\left(\mathcal{A}_0(K)^c\right) \leq \frac{\varepsilon}{8}$$

and then n_0 large enough so that all of the above bounds are valid. Take complements in the above inclusion and consider the first value of n so that $\omega \in \mathcal{A}_{1,n}^c$ or $\mathcal{A}_{2,n}^c$ to see that (recall $\mathcal{A}_{i,n_0-1}$ is the entire space)

$$\begin{aligned}
\mathcal{K}(t_\infty)^c \subset \mathcal{A}_0(K)^c \quad &\cup \quad \left[\bigcup_{n=n_0}^{\infty} \left(\mathcal{A}_{2,n}^c \cap \mathcal{A}_{1,n-1} \cap \mathcal{A}_0(K)\right)\right] \\
&\cup \quad \left[\bigcup_{n=n_0}^{\infty} \left(\mathcal{A}_{1,n}^c \cap \mathcal{A}_{2,n-1} \cap \mathcal{A}_0(K)\right)\right].
\end{aligned}$$

Therefore Lemmas 3.1, 3.2 and 3.3 and our choice of K imply

$$\begin{aligned}
P(\mathcal{K}(t_\infty)^c) &\leq \frac{\varepsilon}{8} + \frac{\varepsilon}{4} + \sum_{n=n_0+1}^{\infty} 2^{n_0-n}\frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&< \varepsilon.
\end{aligned}$$

This proves (3.1), and finishes the proof of the theorem. ∎

4 Proof of Theorems 2 and 3

We first introduce a setting for mutually catalytic branching models. Let $Q = (q_{xy})$ be the Q-matrix for a continuous time $\mathbf{Z}^d$-valued Markov chain ξ_t with semigroup P_t and transition functions $\{p_t(x,y) : t \geq 0, x, y \in \mathbf{Z}^d\}$. If $|x| = |(x_1, \ldots, x_d)| = \sum_{i=1}^d |x_i|$, $(x \in \mathbf{Z}^d)$, we assume the following hypotheses introduced in [DP98]:

$(H1)$ $\|q\|_\infty = \sup_x |q_{xx}| < \infty$.

$(H2)$ For each $x, y \in \mathbf{Z}^d$, $q_{xy} = q_{yx}$ and so $p_t(x,y) = p_t(y,x)$.

$(H3)$ There are increasing positive functions $c(T,\lambda)$ and $\lambda'(\lambda)$ such that
$$\sum_y (|q_{xy}| + p_t(x,y)) \exp(\lambda|y|) \leq c(T,\lambda) \exp(\lambda'(\lambda)|x|)$$
$$\forall t \in [0,T], x \in \mathbf{Z}^d.$$

These conditions are satisfied by a continuous time symmetric random walk with subexponential tail (Lemma 2.1 of [DP98]) and in particular by the nearest neighbor random walk considered in the introduction for which $q_{xy} = \mathbf{1}(|x-y| = 1) - 2d\mathbf{1}(x = y)$. Our generalized mutually catalytic system is then

$$\begin{aligned} U_t(x) &= U_0(x) + \int_0^t QU_s(x)ds + \int_0^t \sqrt{U_s(x)V_s(x)}dB_{1,x}(s), \\ V_t(x) &= V_0(x) + \int_0^t QV_s(x)ds + \int_0^t \sqrt{U_s(x)V_s(x)}dB_{2,x}(s), \qquad (4.1) \\ x \in \mathbf{Z}^d, &\qquad U_0, V_0 \in M_F(\mathbf{Z}^d). \end{aligned}$$

Here, $\{B_{i,x}(t)\}_{x \in \mathbf{Z}^d; i=1,2}$ is a collection of independent $\mathcal{F}_t$-Brownian motions on some filtered probability space. The weak existence and uniqueness of solutions in $C([0,\infty), M_F(\mathbf{Z}^d)^2)$ described in the introduction for $Q = \Delta$ continues to hold and we let P_{U_0,V_0} continue to denote the unique law of the solution on this space of paths.

Theorem 2 continues to hold without change in this more general setting as we now show.

Proof of Theorem 2. By Theorem 2.2(b)(ii) of [DP98] $V_t(x)$ has mean $U_0P_t(x)$ and variance

$$\sum_y \int_0^t p_{t-s}(y,x)^2 U_0P_s(y)V_0P_s(y)ds$$

$$\begin{aligned}
&\le \int_0^t \left(\sum_{y_1} p_{t-s}(y_1,x)U_0P_s(y_1)\right)\left(\sum_{y_2} p_{t-s}(y_2,x)V_0P_s(y_2)\right) ds \\
&= \int_0^t U_0P_t(x)V_0P_t(x)ds = tU_0P_t(x)V_0P_t(x).
\end{aligned}$$

By Chebychev's inequality

$$\begin{aligned}
P_{U_0,V_0}\left(U_t(x) \le \frac{1}{2}U_0P_t(x)\right) &\le P_{U_0,V_0}\left(|U_t(x) - U_0P_t(x)| \ge \frac{1}{2}U_0P_t(x)\right) \\
&\le 4t\frac{V_0P_t(x)}{U_0P_t(x)}.
\end{aligned}$$

By (1.6) if $\varepsilon > 0$ and $t > t_0$, we may choose x_0 so that $4tV_0P_t(x_0)/U_0P_t(x_0) < \varepsilon$ and so

$$P_{U_0,V_0}(\langle U_t, 1\rangle > 0) \ge P_{U_0,V_0}\left(U_t(x_0) > \frac{1}{2}U_0P_t(x_0)\right) > 1 - \varepsilon.$$

Since $\langle U_t, 1\rangle$ is a non-negative martingale this shows that

$$P_{U_0,V_0}\left(\langle U_t, 1\rangle > 0 \quad \forall t > 0\right) = 1.$$

The result follows by symmetry. ∎

It is easy to choose initial conditions satisfying (1.6) for simple symmetric random walk in $\mathbf{Z}^d$.

Proposition 4.1 *Assume $Q = \Delta$ so that $\{\xi_t\}$ is simple symmetric random walk on $\mathbf{Z}^d$ with jump rate $2d$. Suppose there are $m > n$ in $\mathbf{Z}$ such that*

$$\begin{aligned}
U_0\left([m,\infty) \times \mathbf{Z}^{d-1}\right) &= 0, \quad V_0\left([m,\infty) \times \mathbf{Z}^{d-1}\right) > 0 \qquad (4.2)\\
U_0\left((-\infty,n] \times \mathbf{Z}^{d-1}\right) &> 0, \quad V_0\left((-\infty,n] \times \mathbf{Z}^{d-1}\right) = 0.
\end{aligned}$$

Then (1.6) holds and hence

$$P_{U_0,V_0}\left(\langle U_t, 1\rangle\langle V_t, 1\rangle > 0 \quad \forall t > 0\right) = 1.$$

We need an elementary estimate for simple random walk.

Lemma 4.1 *Let $\{\xi_t\}$ be simple symmetric random walk on $\mathbf{Z}^d$. Then*

$$\frac{e^{-td}}{\prod\limits_{i=1}^{d} |x_i|!}t^{|x|} \le P_0(\xi_t = x) \le \frac{1}{\prod_{i=1}^d |x_i|!}t^{|x|}$$

where we recall that $|x| = \sum\limits_1^d |x_i|$.

Proof. Suppose $d = 1$ and ξ_t jumps with rate $\lambda > 0$. Then for $x \geq 0$,

$$
\begin{aligned}
p_t^{(\lambda)}(x) &\equiv P_0(\xi_t = x) \qquad (4.3)\\
&= \sum_{n=0}^{\infty} P(\xi \text{ has } n + x \text{ steps to the right up to time } t)\\
&= \sum_{n=0}^{\infty} \binom{2n+x}{n+x} 2^{-2n-x} e^{-\lambda t} \frac{(\lambda t)^{2n+x}}{(2n+x)!}\\
&= e^{-\lambda t} (\lambda t/2)^x \sum_{n=0}^{\infty} \frac{(\lambda t/2)^{2n}}{n!(n+x)!}\\
&\equiv e^{-\lambda t} (\lambda t/2)^x \sigma(x).
\end{aligned}
$$

Clearly

$$
\begin{aligned}
\frac{1}{x!} \leq \sigma(x) &\leq \frac{1}{x!} \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-2n} \frac{(\lambda t)^{2n}}{(2n)!}\\
&\leq \frac{1}{x!} \sum_{n=0}^{\infty} \frac{(\lambda t)^{2n}}{(2n)!} \leq \frac{e^{\lambda t}}{x!}.
\end{aligned}
$$

Put this into (4.3) and use symmetry in x to get

$$
e^{-\lambda t} \frac{(\lambda t/2)^{|x|}}{|x|!} \leq p_t^{(\lambda)}(x) \leq \frac{(\lambda t/2)^{|x|}}{|x|!} \qquad \forall x \in \mathbf{Z}. \qquad (4.4)
$$

Since $P_0(\xi_t = x) = \prod_{i=1}^d p_t^{(1)}(x_i)$, the result follows. ∎

Proof of Proposition 4.1. Choose $v \in \mathbf{Z}^d$ with $v_1 \geq m$ such that $V_0(v) > 0$. Then (4.2) shows that for $t > 2$, $x = (x_1, v_2, \ldots, v_d)$ and $x_1 > v_1$,

$$
\begin{aligned}
\frac{U_0 P_t(x)}{V_0 P_t(x)} &\leq \sum_y \mathbf{1}(y_1 < m) U_0(y) \frac{t^{\sum_1^d (|y_i - x_i| - |v_i - x_i|)}}{\prod_1^d |y_i - x_i|! V_0(v) e^{-dt}} \prod_1^d |v_i - x_i|!\\
&\leq \frac{e^{2dt}}{V_0(v)} (x_1 - v_1)! \sum_{y_1 < m} \frac{t^{v_1 - y_1}}{|y_1 - x_1|!} \sum_{y_2, \ldots, y_d} U_0(y) \frac{t^{\sum_2^d |y_i - x_i|}}{\prod_2^d |y_i - x_i|!}\\
&\leq \frac{e^{3dt}}{V_0(v)} \sum_{y_1 < m} \frac{t^{(v_1 - m) + (m - y_1)}}{(x_1 - y_1) \ldots (x_1 - v_1 + 1)} U_0^{(1)}(y_1),
\end{aligned}
$$

where $U_0^{(1)}(y_1) = \sum_{y_2 \dots y_d} U_0(y_1, y_2, \dots, y_d)$ is the first marginal of U_0. Setting $k = m - y_1$, we see that

$$\begin{aligned}
\frac{U_0 P_t(x)}{V_0 P_t(x)} &\leq t^{v_1 - m} \frac{e^{3dt}}{V_0(v)} \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot \frac{k!}{(x_1 - m + k) \dots (x_1 - v_1 + 1)} \\
&\quad \cdot U_0^{(1)}(m-k) \\
&\leq c(t, v, m) e^{(3d+1)t} \sum_{k=1}^{\infty} \frac{U_0^{(1)}(m-k)}{(x_1 - v_1 + 1)} \\
&\leq \frac{c'(t, v, m) \langle U_0, 1 \rangle}{x_1 - v_1 + 1} \\
&\to 0 \quad \text{as} \quad x_1 \to \infty.
\end{aligned}$$

This, and a symmetrical argument for the reciprocal, establish (1.6) and we are done. ∎

Next, we turn to the main task of this section, proving Theorem 3 in our more general Markov chain setting. In this case we need to add a pair of hypotheses.

Theorem 4 *Assume*

$$(H3) \text{ holds with } \lambda'(\lambda) = \lambda \tag{4.5}$$

and

$$\inf_{s \leq t, x \in \mathbf{Z}^d} p_s(x, x) = \varepsilon_0(t) > 0 \tag{4.6}$$

Under the hypotheses on V_0 in Theorem 3, the conclusion of that result holds.

Remark. The hypotheses added above hold for any continuous time random walk with subexponential jump distributions, as (4.6) is trivial and (4.5) is proved in Lemma 2.1 of [DP98].
Notation. $\phi_\lambda(x) = e^{\lambda |x|}$ for $x \in \mathbf{Z}^d$.

Lemma 4.2 *Assume the hypotheses of Theorem 4. Then $\forall \lambda > 0$, $\varepsilon > 0$, $T > 0$ there is a $C_{T,\varepsilon,\lambda} > 0$ such that*

$$\varepsilon_0(T) \phi_{-\lambda}(x) \leq P_t \phi_{-\lambda}(x) \leq C_{T,\varepsilon,\lambda} \phi_{-\lambda+\varepsilon}(x) \qquad \forall t \leq T.$$

Proof. For the lower bound, observe that $P_t\phi_{-\lambda}(x) \geq p_t(x,x)\phi_{-\lambda}(x) \geq \varepsilon_0(T)\phi_{-\lambda}(x)$ for all $t \leq T$ by (4.6).

For the upper bound, note that (H_2) implies

$$p_t(x,y) = p_t(y,x) \leq C_{T,\lambda-\varepsilon}e^{(\lambda-\varepsilon)|y|-(\lambda-\varepsilon)|x|} \quad \text{for} \quad t \leq T$$

and so for $t \leq T$

$$P_t(\phi_{-\lambda})(x) \leq C_{T,\lambda-\varepsilon}\sum_{y\in\mathbf{Z}^d} e^{-\varepsilon|y|}e^{-(\lambda-\varepsilon)|x|} \leq C_{T,\varepsilon,\lambda}\phi_{-(\lambda-\varepsilon)}(x).$$

This finishes the proof of Lemma 4.2. ∎

Proof of Theorem 4. The hypotheses on $\{\lambda_i\}$ allow us to choose β such that

$$2\lambda_1 - \Big(\frac{\lambda_0+\lambda_2}{2}\Big) < \beta < \lambda_2 \tag{4.7}$$

and then α such that

$$2\lambda_1 - \beta < \alpha < (\lambda_0+\lambda_1)/2. \tag{4.8}$$

For now we fix $\eta \in (0,1]$ (recall $U_0 \in \eta\phi_{-\lambda_0}$)and will specify its value later in the proof. It is easy to modify the derivation of Theorem 2.2(c) of [DP98] to see that

$$\langle V_t, \phi_\beta\rangle = \langle V_0, \phi_\beta\rangle + \int_0^t \langle V_s, Q\phi_\beta\rangle ds + M_t^V(\phi_\beta), \tag{4.9}$$

where $M_t^V(\phi_\beta)$ is a continuous square integrable martingale such that

$$\langle M^V(\phi_\beta)\rangle_t = \int_0^t \langle U_sV_s, \phi_\beta^2\rangle ds.$$

To see this note that $\beta < \lambda_2$ shows that $\langle V_0, \phi_\beta\rangle < \infty$, and that (4.5) implies

$$|Q\phi_\beta| \leq c(\beta)\phi_\beta \tag{4.10}$$

so that

$$\begin{aligned} E\left(\langle V_s, |Q\phi_\beta|\rangle\right) &\leq c(\beta)\langle V_0, P_s\phi_\beta\rangle \\ &\leq c(\beta)C_{T,\beta}\langle V_0, \phi_\beta\rangle \qquad \text{(by (4.5))} \\ &< \infty. \end{aligned} \tag{4.11}$$

In addition we use the fact that

$$
\begin{aligned}
E\left(\int_0^t \langle U_s V_s, \phi_\beta^2\rangle ds\right) &= \int_0^t \sum_x e^{2\beta|x|} U_0 P_s(x) V_0 P_s(x) ds \\
&\qquad \text{(Theorem 2.2 of [DP98])} \\
&\le \int_0^t \sum_x e^{2\beta|x|} c_{t,\varepsilon,\lambda_0} \eta e^{-(\lambda_0-\varepsilon)|x|} c_{t,\varepsilon,\lambda_2} c_2 e^{-(\lambda_2-\varepsilon)|x|} ds \\
&\qquad \text{(Lemma 4.2)} \\
&< \infty \qquad \text{for any} \quad t > 0,
\end{aligned}
$$

for $\varepsilon > 0$ small enough because $2\beta < 2\lambda_2 < \lambda_0 + \lambda_2$. (4.11) and the above show that

$$
V_T^{(\beta)} \equiv \sup_{t\le T} \langle V_t, \phi_\beta\rangle \in L^1 \text{ and } \sup_{0<\eta\le 1} P_{U_0,V_0}\left(V_T^{(\beta)}\right) \le C_T^V < \infty. \tag{4.12}
$$

A similar argument (now use $\alpha < (\lambda_0 + \lambda_1)/2 < \lambda_0$) shows that

$$
U_T^{(\alpha)} = \sup_{t\le T} \langle U_t, \phi_\alpha\rangle \in \mathbf{L}^1 \text{ and } \sup_{0<\eta\le 1} P_{U_0,V_0}\left(U_T^{(\alpha)}\right) \le C_T^U < \infty. \tag{4.13}
$$

Now fix $\varepsilon > 0$. We claim there is a $t_0 > 0$, independent of the choice of $\eta \in (0, 1]$, such that

$$
P_{U_0,V_0}\left(V(t,x) \ge \frac{1}{2} V_0 P_t(x) \quad \forall t \le t_0 \quad \forall x \in \mathbf{Z}^d\right) > 1 - \varepsilon. \tag{4.14}
$$

If $N(t,x) = V(t,x) - V_0 P_t(x)$, then Theorem 2.2(b) of [DP98] shows that if $0 \le t \le u \le 1$ then $N(u,x) - N(t,x) = N_{t,x}^{(1)}(u) + N_{u,x}^{(2)}(t)$, where

$$
N_{t,x}^{(1)}(u) = \sum_y \int_t^u p_{u-s}(y,x) \sqrt{U_s(y) V_s(y)} dB_{2,y}(s)
$$

and

$$
N_{u,x}^{(2)}(t) = \sum_y \int_0^t (p_{u-s}(y,x) - p_{t-s}(y,x)) \sqrt{U_s(y) V_s(y)} dB_{2,y}(s).
$$

Our hypotheses on $\{\xi_t\}$ imply that

$$
|p_r(x,y) - p_s(x,y)| = \left|\int_s^r Q p_w(\cdot, y)(x) dw\right| \le \|q\|_\infty |r - s|. \tag{4.15}
$$

We have for any $\delta > 0$,

$$\begin{aligned}
\langle N^{(2)}_{u,x}\rangle_t &= \sum_y \int_0^t \left(p_{u-s}(y,x) - p_{t-s}(y,x)\right)^2 U_s(y)V_s(y)ds && (4.16)\\
&\le \|q\|_\infty (u-t)\int_0^t \sum_y \left(p_{u-s}(y,x) + p_{t-s}(y,x)\right)\\
&\quad \cdot U_1^{(\alpha)} V_1^{(\beta)} \phi_{-\alpha}(y)\phi_{-\beta}(y)ds\\
&\qquad\qquad\qquad\qquad \text{(by (4.15), (4.12) and (4.13))}\\
&\le U_1^{(\alpha)} V_1^{(\beta)} \|q\|_\infty (u-t) C_{1,\delta,\alpha+\beta}\phi_{-\alpha-\beta+\delta}(x) \quad \text{(Lemma 4.2)}\\
&\equiv c(\delta) U_1^{(\alpha)} V_1^{(\beta)} (u-t)\phi_{-\alpha-\beta+\delta}(x)
\end{aligned}$$

and, using similar reasoning,

$$\begin{aligned}
\langle N^{(1)}_{t,x}\rangle_u &= \sum_y \int_t^u p_{u-s}(y,x)^2 U_s(y)V_s(y)ds\\
&\le U_1^{(\alpha)} V_1^{(\beta)} \int_t^u \sum_y p_{u-s}(x,y)\phi_{-\alpha}(y)\phi_{-\beta}(y)ds\\
&\le U_1^{(\alpha)} V_1^{(\beta)} (u-t)\sup_{s\le u} P_s\phi_{-\alpha-\beta}(x)\\
&\le c(\delta) U_1^{(\alpha)} V_1^{(\beta)} (u-t)\phi_{-\alpha-\beta+\delta}(x).
\end{aligned}$$

Choose $\delta > 0$ small enough so that (see (4.8))

$$2\lambda \equiv \alpha + \beta - \delta > 2\lambda_1. \tag{4.17}$$

Let $\Delta(n,x) = (n+1)^{1/2}2^{-n/2}(|x|+1)^{1/2}\phi_{-\lambda}(x)$, $x \in \mathbf{Z}^d$, $n \in \mathbf{N}$. Then for $K, K_1 > 0$

$$\begin{aligned}
&P_{U_0,V_0}\left(\left|N\left(\frac{j+1}{2^n},x\right) - N\left(\frac{j}{2^n},x\right)\right| \ge K\Delta(n,x), \quad U_1^{(\alpha)}V_1^{(\beta)} \le K_1^2\right)\\
&\le P_{U_0,V_0}\Big(\left|N^{(1)}_{j/2^n,x}\left((j+1)/2^n\right)\right| \ge \frac{K}{2}\Delta(n,x),\\
&\qquad \langle N^{(1)}_{j2^{-n},x}\rangle \left((j+1)2^{-n}\right) \le c(\delta)K_1^2 2^{-n}\phi_{-2\lambda}(x)\Big)\\
&\quad +P_{U_0,V_0}\Big(\left|N^{(2)}_{(j+1)2^{-n},x}\left(j2^n\right)\right| \ge \frac{K}{2}\Delta(n,x),\\
&\qquad \langle N^{(2)}_{(j+1)2^{-n},x}\rangle (j2^{-n}) \le c(\delta)K_1^2 2^{-n}\phi_{-2\lambda}(x)\Big)\\
&\le 2P\left(\sup_{s\le 1}|B_s| > \frac{1}{2}K\Delta(n,x)\left(c(\delta)K_1^2 2^{-n}\phi_{-2\lambda}(x)\right)^{-1/2}\right),
\end{aligned}$$

where B is a one-dimensional Brownian motion and we have used the Dubins-Schwarz Theorem in the last line. An elementary estimate on the Gaussian tail and the fact that

$$\begin{aligned}
&\frac{1}{2}K\Delta(n,x)\Big(c(\delta)K_1^2 2^{-n}\phi_{-2\lambda}(x)\Big)^{-1/2} \\
&\qquad = KK_1^{-1}(n+1)^{1/2}(|x|+1)^{1/2}\left(2\sqrt{c\,(\delta)}\right)^{-1}
\end{aligned}$$

shows that if we set $L = K^2/4K_1^2c(\delta)$ and assume $L \geq 1$, then we have

$$\begin{aligned}
P_{U_0,V_0}\Bigg(&\left|N\left(\frac{j+1}{2^n},x\right) - N\left(\frac{j}{2^n},x\right)\right| \geq K\Delta(n,x) \\
&\qquad \text{for some} \quad 0 \leq j < 2^n, x \in \mathbf{Z}^d \quad \text{and} \quad n \in \mathbf{N}\Bigg) \\
\leq\ & 8\sum_{n=1}^{\infty}\sum_{x\in\mathbf{Z}^d} 2^n \exp\left(\frac{-K^2(n+1)(|x|+1)}{4K_1^2c(\delta)}\right) + P_{U_0,V_0}\left(U_1^{(\alpha)}V_1^{(\beta)} > K_1^2\right) \\
\leq\ & 4\sum_{x\in\mathbf{Z}^d}\sum_{n=1}^{\infty} \exp\left\{-\left[\frac{K^2(|x|+1)}{4K_1^2c(\delta)} - \log 2\right](n+1)\right\} \\
& + P_{U_0,V_0}\left(U_1^{(\alpha)} > K_1\right) + P_{U_0V_0}\left(V_1^{(\beta)} > K_1\right) \\
\leq\ & c\sum_{x\in\mathbf{Z}^d} \exp\left\{-L(|x|+1)\right\} + (K_1)^{-1}\left(C_1^U + C_1^V\right),
\end{aligned}$$

where we have used (4.12) and (4.13) in the last line. First choose K_1 and then K sufficiently large so that the above expression is less than ε (and $L \geq 1$). Note that the choice of K_1 and K may be made independently of $\eta \in (0,1]$. Therefore off a set of P_{U_0,V_0}-measure at most ε if $2^{-n_0} \leq t < 2^{1-n_0}$ ($n_0 \in \mathbf{N}$) and $t = \sum_{n_0}^{\infty} j_n 2^{-n}$ where $j_{n_0} = 1$ and $j_n \in \{0,1\}$ for $n > n_0$, then for all x in $\mathbf{Z}^d$

$$\begin{aligned}
|N(t,x)| &\leq \left(\sum_{n=n_0}^{\infty} j_n 2^{-n/2}(n+1)^{1/2}\right) K(|x|+1)^{1/2}\phi_{-\lambda}(x) \\
&\leq cK(t\log 1/t)^{1/2}(|x|+1)^{1/2}\phi_{-\lambda}(x)\varepsilon_0(1)^{-1}c_1^{-1}\phi_{\lambda_1}(x)P_tV_0(x) \\
&\qquad\qquad \text{(by the lower bound in Lemma 4.2)} \\
&\leq cK(t\log 1/t)^{1/2}P_tV_0(x) \qquad \text{(since } \lambda_1 < \lambda\text{)}.
\end{aligned}$$

Hence we may choose $t_0 > 0$ sufficiently small (independent of $\eta \in (0,1]$ and c_0) such that

$$P_{U_0,V_0}\left(|N(t,x)| \leq \frac{1}{2}P_tV_0(x) \qquad \text{for all} \quad 0 \leq t \leq t_0, x \in \mathbf{Z}^d\right) > 1-\varepsilon.$$

This proves (4.14).

Let

$$\sigma = \inf\left\{t : V_t(x) < \frac{1}{2}P_tV_0(x) \quad \text{for some} \quad x \in \mathbf{Z}^d\right\} \wedge 1.$$

As for (4.9) we have

$$\langle U_t, \phi_\alpha\rangle = \langle U_0, \phi_\alpha\rangle + \int_0^t \langle U_s, Q\phi_\alpha\rangle ds + M_t^U(\phi_\alpha) \tag{4.18}$$

where $M_t^U(\phi_\alpha)$is a continuous square-integrable martingale such that for $t \leq \sigma$,

$$\begin{aligned}\frac{d}{dt}\langle M^U(\phi_\alpha)\rangle &= \langle U_tV_t, \phi_\alpha^2\rangle \\ &\geq \frac{1}{2}\langle U_tP_tV_0, \phi_\alpha^2\rangle \\ &\geq \frac{c_1}{2}\varepsilon_0(1)\langle U_t, \phi_\alpha\rangle.\end{aligned}$$

In the last line we use Lemma 4.2 and the fact that $\alpha > \lambda_1$ by (4.7) and (4.8). Therefore

$$\frac{d}{dt}\langle M^U(\phi_\alpha)\rangle_t \geq c_{4.19}\langle U_t, \phi_\alpha\rangle \qquad \text{for} \quad t \leq \sigma. \tag{4.19}$$

Let

$$\begin{aligned}C(t) &= \int_0^t \langle U_s, \phi_\alpha\rangle^{-1} d\langle M^U(\phi_\alpha)\rangle_s, \\ \tau(t) &= \inf\{u : C(u) > t\} \qquad (\inf \phi = \infty),\end{aligned}$$

and

$$\tilde{U}(t) = \langle U_{\tau(t)}, \phi_\alpha\rangle \qquad \text{for} \quad t < C(\infty).$$

Then for $t < C(\sigma)$

$$\begin{aligned}\tilde{U}(t) &= \langle U_0, \phi_\alpha\rangle + \int_0^{\tau(t)} \langle U_s, Q\phi_\alpha\rangle ds + \tilde{M}(t) \\ &= \langle U_0, \phi_\alpha\rangle + \int_0^t \langle U_{\tau_r}, Q\phi_\alpha\rangle \tau'(r) dr + \tilde{M}(t)\end{aligned}$$

where $\tilde{M}(t) = M^U_{\tau(t)\wedge\sigma}(\phi_\alpha)$ is a continuous local martingale such that for $t < C(\sigma)$

$$\frac{d}{dt}\langle\tilde{M}\rangle_t = \langle M^U(\phi_\alpha)\rangle'(\tau(t))\tau'(t) = \langle M^U(\phi_\alpha)\rangle'(\tau(t))C'(\tau(t))^{-1} = \tilde{U}(t).$$

Hence by enlarging the probability space if necessary we may assume there is a filtration $(\tilde{\mathcal{F}}_t)$ and an $(\tilde{\mathcal{F}}_t)$-Brownian motion, $B(t)$, such that $U_{\tau(t)}$, $\tilde{U}_t$ and $\tau'(t)\mathbf{1}(t < C(\sigma))$ are $(\tilde{\mathcal{F}}_t)$-adapted, $C(\sigma)$ is an $(\tilde{\mathcal{F}}_t)$-stopping time and

$$\tilde{U}(t) = \langle U_0, \phi_\alpha\rangle + \int_0^t \langle U_{\tau_r}, Q\phi_\alpha\rangle\tau'(r)dr + \int_0^t \sqrt{\tilde{U}(r)}dB_r,\ t < C(\sigma). \tag{4.20}$$

(4.19) implies that $\tau'(r) \le c_{4.19}^{-1}$ for $r < C(\sigma)$ and the analogue of (4.10) now shows that for $r < C(\sigma)$,

$$\begin{aligned}\left|\langle\tilde{U}_r, Q\phi_\alpha\rangle\tau'(r)\right| &\le c(\alpha)c_{4.19}^{-1}\tilde{U}_r \\ &\equiv c_{4.21}\tilde{U}_r.\end{aligned}$$

Let $\hat{U}_t$ be the pathwise unique solution of

$$\hat{U}(t) = \langle U_0, \phi_\alpha\rangle + \int_0^t c_{4.21}\hat{U}_r dr + \int_0^t \sqrt{\hat{U}(r)}dB(r). \tag{4.21}$$

A comparison theorem for stochastic differential equations (see Rogers and Williams [RW87], V.43.1) shows that

$$\tilde{U}(t) \le \hat{U}(t) \qquad \text{for} \quad t < C(\sigma)$$

and so

$$\langle U_t, \phi_\alpha\rangle \le \hat{U}(C(t)) \qquad \text{for} \quad t < \sigma. \tag{4.22}$$

(4.19) shows that $C(t) \ge c_{4.19}t$ for $t \le \sigma$ and so if P_x is the law of $\hat{U}$ starting at x we have for $0 < t_1 \le t_0$ (here $\varepsilon > 0$ is fixed and t_0 is as in (4.14))

$$\begin{aligned}P_{U_0,V_0}(U_t = 0 \quad \forall t \ge t_1) &= P_{U_0,V_0}(U_{t_1} = 0) \\ &\ge P\left(\hat{U}(C(t_1)) = 0, \quad t_1 < \sigma\right) \qquad \text{(by (4.22))} \\ &\ge P_{\eta\langle\phi_{-\lambda_0},\phi_\alpha\rangle}\left(\hat{U}(c_{4.19}t_1) = 0\right) - P_{U_0,V_0}(\sigma \le t_0) \\ &\ge P_1\left(\hat{U}(c_{4.19}t_1) = 0\right)^{\eta\langle\phi_{-\lambda_0},\phi_\alpha\rangle} - \varepsilon \text{ (by (4.14))},\end{aligned}$$

where we have used the multiplicative property of the superprocess $\hat{U}$ and the choice of t_0. Now $P_1(\hat{U}(c_{4.19}t_1) = 0) > 0$ (in fact it is easy to get an explicit expression for this probability, or alternatively one may use Girsanov's theorem and the fact that this probability is positive if $c_{4.21} = 0$ in (4.21). Therefore for $\eta > 0$ sufficiently small the above probability is at least $1 - 2\varepsilon$. This completes the proof of Theorem 4. ∎

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